JOURNAL OF DIFFERENTIAL EQUATIONS 48, 35–59 (1983)

Symplectic Structure and Conserved Quantities for Some New Conformally Covariant Systems

THOMAS P. BRANSON*

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

Received July 20, 1981; revised January 28, 1982

Let M be the n-dimensional Minkowski space, $n \geqslant 3$. One consequence of [1] is that the null space of the equation $\{(n-2k+2)\, d^*d + (n-2k-2)\, dd^*\}\, \Phi = 0$ on differential k-forms Φ in M is conformally covariant. The same is true of a nonlinear equation obtained by adding to the above a term homogeneous of degree $(n+2)/(n-2)$. This generalizes the well-known conformal covariance properties of the wave equation and the equations $\Box\phi \pm \phi^{(n+2)/(n-2)} = 0$ when $k = 0$, and of Maxwell's equations on a vector potential when $k = (n \pm 2)/2$ (and n is even). We define a natural (conformally invariant) symplectic structure for the new equations, and use it to calculate the $(n+1)(n+2)/2$ conserved quantities corresponding to the standard conformal group generators.

1. Introduction

In [1], it is shown that each pseudo-Riemannian manifold of dimension $n \geqslant 3$ possesses a distinguished second-order linear differential operator L on k-forms, $0 \leqslant k \leqslant n$, with unusual conformal covariance properties. If d is the exterior derivative and δ the coderivative (defined below), calculated with respect to the metric tensor, we set $L' = (n-2k+2)\, \delta d + (n-2k-2)\, d\delta$. L is then L' plus a zeroth order term Z whose components depend on the Ricci tensor. (Z depends only on the scalar curvature if $k = 0$, and $Z = 0$ if $k = (n \pm 2)/2$.) A distinguished nonlinearity, essentially the $(n+2)/(n-2)$ power in a sense defined below, may be added to L without disturbing conformal covariance.

If M is the n-dimensional flat Minkowski space, Z vanishes and we get the wave equation and the nonlinear equations $\Box\phi \pm \phi^{(n+2)/(n-2)} = 0$ for $k = 0$, and Maxwell's equations (on a vector potential) when $k = (n \pm 2)/2$. Since the conformal group of M is $(n+1)(n+2)/2$-dimensional, one expects $(n+1)(n+2)/2$ independent conserved quantities ("Noether currents") for the new equations; the object of this paper is to calculate them.

* Research supported by NSF grant MCS 8005235.

0022-0396/83 $3.00

In Section 2, we discuss the conformal covariance of the equations more precisely, write the equations in evolution form, as is appropriate to the Cauchy problem (for which existence and uniqueness is not treated here), and explicitly write the $(n+1)(n+2)/2$ new solutions, gotten as infinitesimal conformal transforms of a given solution. In Section 3, we review a few facts from the symplectic geometry of Hamiltonian systems in finitely many degrees of freedom. These are formally applied in Section 4 to our system (which, of course, has infinitely many degrees of freedom) to (heuristically) derive formulas for the conserved quantities in terms of Cauchy data at a fixed time. This requires the choice of a symplectic structure which is *conformally invariant*, that is, in which the formal vector fields in Cauchy data space corresponding to infinitesimal conformal transformations are (formally) Hamiltonian. Our choice (4.1) agrees with the standard symplectic structures for the wave and Maxwell equations (Remarks 4.1 and 7.1).

In Section 5 we justify the heuristics of Section 4 by showing that the quantities calculated there are indeed conserved by the equations, with appropriate decay assumptions on Cauchy data. In Section 6 it is shown how the analysis of Sections 4 and 5 can be modified to give conservation laws for the nonlinear equations, in analogy with (and generalizing) results [5] for $\Box\phi \pm \phi^{(n+2)/(n-2)} = 0$.

Before proceeding, we fix some geometric notation. If T is a vector field in $\mathbb{R}^n$, we denote by L_T the *Lie derivative* (of tensor fields) with respect to T, and by $\iota(T)$ *interior multiplication* of differential forms by T:

$$(\iota(T)\,\omega)(X_1,\dots,X_{k-1}) = \omega(T, X_1,\dots,X_{k-1}),$$

where the X_i are vector fields, and ω is a k-form. Recall (see, e.g., [7]) that

$$L_T\omega = \iota(T)\,d\omega + d\iota(T)\,\omega. \tag{1.1}$$

Now suppose $\mathbb{R}^n$ is equipped with the *standard pseudometric* g of signature (p,q), $p+q=n$: if $\varepsilon_1 = \cdots = \varepsilon_p = -\varepsilon_{p+1} = \cdots = -\varepsilon_n = 1$, $X = \sum_{i=1}^n X^i\partial_i$, $Y = \sum_{i=1}^n Y^i\partial_i$, $\partial_i = \partial/\partial x^i$, then

$$g(X,Y) = \sum_{i=1}^{n} \varepsilon_i X^i Y^i.$$

We denote by $\mathbb{R}^{(p,q)}$ the pair $(\mathbb{R}^n, g)$. A vector field T on $\mathbb{R}^{(p,q)}$ is *conformal* if $L_T g = \rho_T g$ for some C^∞ function ρ_T.

On $\mathbb{R}^{(p,q)}$, with the usual identification of vector fields with one-forms through g, we get a nondegenerate inner product g^1 on one-forms, and a nondegenerate inner product g^k on k-forms characterized by

$$g^k(\omega^1\wedge\cdots\wedge\omega^k, \eta^1\wedge\cdots\wedge\eta^k) = \det(g^1(\omega^i,\eta^j)),$$

where ω^i and η^j are one-forms. Fixing the orientation $E = dx^n \wedge \cdots \wedge dx^1$, we define the Hodge star operator $*$ carrying k-forms to $(n-k)$-forms by

$$g^{n-k}(*\,\omega, \eta) = g^n(\omega \wedge \eta, E),$$

where ω is a k-form, and η an $(n-k)$-form. The *coderivative* δ lowering orders of forms by one is given on k-forms by $(-1)^{n(k+1)+1+q} *d*$. δ is the formal adjoint of d acting on $(k-1)$-forms:

$$\int_{\mathbb{R}^n} g^k(d\omega, \eta)\, E = \int_{\mathbb{R}^n} g^{k-1}(\omega, \delta\eta)\, E$$

if ω is a $(k-1)$-form and η a k-form, at least one of which has compact support. Indeed, $g^k(d\omega, \eta) - g^{k-1}(\omega, \delta\eta) = d(\omega \wedge *\eta)$, so by Stokes' Theorem, the integral of $g^k(d\omega, \eta) - g^{k-1}(\omega, \delta\eta)$ vanishes if $\omega \wedge *\eta$ vanishes at infinity.

The *d'Alembertian* $\square_k$ on k-forms (or *Laplacian* Δ_k if $q = 0$) is $\delta d + d\delta$; its effect is to operate on each component of a form by $\square_0 = -\partial_1^2 - \cdots - \partial_p^2 + \partial_{p+1}^2 + \cdots + \partial_n^2$.

Finally, if T is a vector field and $\tilde{T}$ the corresponding one-form, $\iota(T)$ is the (pointwise) adjoint of the *exterior multiplication* $\varepsilon(\tilde{T})\colon \phi \mapsto \tilde{T} \wedge \phi$, regardless of the metric signature: $g^k(\varepsilon(\tilde{T})\,\phi, \psi) = g^{k-1}(\phi, \iota(T)\,\psi)$. (The metric comes in in calculating $\tilde{T}$.) A short calculation gives $\iota(T) = (-1)^{n(k+1)+q} * \varepsilon(\tilde{T})*$ on k-forms.

2. The Equations and their Group Invariance

In this section, T is always a conformal vector field on the n-dimensional Minkowski space $\mathbb{R}^{(n-1,1)}$. According to [1], the operator on k-forms

$$\tilde{\square}_k = \delta d + \beta d\delta, \qquad \beta = \frac{n-2k-2}{n-2k+2}, \qquad k \neq \frac{n+2}{2},$$

is conformally covariant in the sense that

$$\tilde{\square}_k \Phi = 0 \Rightarrow \tilde{\square}_k \left(L_T + \frac{n-2k-2}{4}\rho_T \right) \Phi = 0. \tag{2.1}$$

When $k = 0$, this is the familiar covariance result for the wave equation; when n is even and $k = (n-2)/2$, β vanishes and we get the *Maxwell equations* $\delta d\Phi = 0$ as conditions on a *vector potential* Φ. (The *field strengths* $\Omega = d\Phi$ satisfy what are usually called the Maxwell equations: $d\Omega = 0$, $\delta\Omega = 0$.)

In analogy with the conformal covariance of $\Box\phi + \alpha\phi^{(n+2)/(n-2)} = 0$ on functions, we have a covariance result for $\tilde{\Box}_k \Phi + \alpha\Phi^{((n+2)/(n-2))} = 0$ on k-forms, where

$$\Phi^{(r)} = g^k(\Phi, \Phi)^{(r-1)/2}\, \Phi, \; r \text{ rational}.$$

Let $s(\Phi) = \tilde{\Box}_k \Phi + \alpha\Phi^{((n+2)/(n-2))}$, and let S_Φ be the linearization of s at Φ:

$$S_\Phi \Omega = \tilde{\Box}_k \Omega + \alpha g^k(\Phi, \Phi)^{2/(n-2)}\, \Omega + \alpha \frac{4}{n-2} g^k(\Phi, \Phi)^{(4-n)/(n-2)}\, g^k(\Phi, \Omega)\, \Phi.$$

Then

$$s(\Phi) = 0 \Rightarrow S_\Phi \left(L_T + \frac{n-2k-2}{4} \rho_T\right) \Phi = 0. \tag{2.2}$$

For functions,

$$\Box\phi + \alpha\phi^{(n+2)/(n-2)} = 0 \Rightarrow \left(\Box + \alpha \frac{n+2}{n-2} \phi^{4/(n-2)}\right)\left(L_T + \frac{n-2}{4} \rho_T\right) \phi = 0.$$

To begin the process of deriving and verifying the conservation laws associated to (2.1) and (2.2), we write the equations in evolution form. Let $t = x^n$ and denote $L_{\partial/\partial t}$ by a circumscribed dot. A typical k-form Φ on $\mathbb{R}^{(n-1,1)}$ may be written $\Phi = dt \wedge \Phi_0 + \Phi_1$, where Φ_0 and Φ_1 are time-dependent forms on the Euclidean space $\mathbb{R}^{(n-1,0)} = \mathbb{R}^{n-1}$.

We concentrate for now on the linear equations; all results obtained will be modified in Section 6 to apply to the nonlinear equations. If we let $d^{(n)}$, $\delta^{(n)}$, $*^{(n)}$ be the exterior derivative, coderivative, and Hodge star in $\mathbb{R}^{(n-1,1)}$, with d, δ, $*$ denoting the similar objects in $\mathbb{R}^{n-1}$, we get

$$\begin{aligned}
d^{(n)}\Phi &= dt \wedge (\dot{\Phi}_1 - d\Phi_0) + d\Phi_1, \\
*^{(n)}\Phi &= dt \wedge (-1)^k *\Phi_1 - *\Phi_0, \\
\delta^{(n)}\Phi &= -dt \wedge \delta\Phi_0 + \dot{\Phi}_0 + \delta\Phi_1.
\end{aligned}$$

Thus

$$\begin{aligned}
\tilde{\Box}_k \Phi = dt \wedge \{&\beta\ddot{\Phi}_0 + D\Phi_0 - (1-\beta)\, \delta\dot{\Phi}_1\} \\
&+ \{\ddot{\Phi}_1 + D\Phi_1 - (1-\beta)\, d\dot{\Phi}_0\},
\end{aligned}$$

where $D = \delta d + \beta d\delta$ on all orders of forms in $\mathbb{R}^{n-1}$. (β is fixed at $(n-2k-2)/(n-2k+2)$ and is *not* adjusted for the change in dimension from n to $n-1$, or for changes in the order of form.) Writing Ψ for $\dot{\Phi}$, the equation $\tilde{\Box}_k \Phi = 0$ becomes

$$\dot{\Psi}_0 = -\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta} \delta\Psi_1,$$

$$\dot{\Psi}_1 = -D\Phi_1 + (1-\beta)\, d\Psi_0. \tag{2.3}$$

(We require here that $\beta \neq 0$; i.e., $k \neq (n-2)/2$. Thus we exclude the Maxwell equations, the Cauchy problem for which must be treated differently.) When $k=0$, all zero-subscripted components vanish, and we have $\psi = \dot{\phi}$, $\dot{\psi} = -\Delta\phi$.

The conformal vector fields on $\mathbb{R}^{(n-1,1)}$ form an $(n+1)(n+2)/2$-dimensional Lie subalgebra of the Lie algebra of vector fields, and a basis (see, e.g., [4]) is (setting $R = \sum_{i=1}^{n-1} x^i \partial_i$ and $r^2 = \sum_{i=1}^{n-1} (x^i)^2$)

$$\begin{aligned}
&T = \frac{\partial}{\partial t} && (\rho_T = 0),\\
&P_i = \partial_i && (\rho_{P_i} = 0),\\
&\Lambda_{ij} = x^i \partial_j - x^j \partial_i && (\rho_{\Lambda_{ij}} = 0),\\
&B_i = x^i \frac{\partial}{\partial t} + t\partial_i && (\rho_{B_i} = 0),\\
&H = t\frac{\partial}{\partial t} + R && (\rho_H = 2)\\
&I_n = (r^2 + t^2)\frac{\partial}{\partial t} + 2tR && (\rho_{I_n} = 4t),\\
&I_i = -2tx^i H + (r^2 - t^2)\,\partial_i && (\rho_{I_i} = -4x^i).
\end{aligned} \tag{2.4}$$

Suppose $\tilde{\Box}_k \Phi = 0$, and denote the Cauchy data of Φ at time t by (Φ, Ψ) (with the usual abuse of notation). (Φ, Ψ) then solves (2.3). According to (2.1) and (2.4), (a)–(g) following are also in the null space of $\tilde{\Box}_k$. (If X is a vector field and ω a form in $\mathbb{R}^n$, let $X\omega$ be the result of applying X to each component of ω. Generally $X\omega \neq L_X \omega$.)

(a) $\dot{\Phi}$, for which the Cauchy data at time t are

$$(Z_0, Z_1, W_0, W_1) = \left(\Psi_0, \Psi_1, -\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta}\delta\Psi_1, -D\Phi_1 + (1-\beta)\, d\Psi_0\right).$$

(b) $\partial_i \Phi$, for which the Cauchy data are

$$(Z_0, Z_1, W_0, W_1) = (\partial_i \Phi_0, \partial_i \Phi_1, \partial_i \Psi_0, \partial_i \Psi_1).$$

(c) $L_{\Lambda_{ij}} \Phi$, for which the Cauchy data are

$$Z_0 = x^i \partial_j \Phi_0 - x^j \partial_i \Phi_0 + dx^i \wedge \iota(\partial_j)\, \Phi_0 - dx^j \wedge \iota(\partial_i)\, \Phi_0,$$

$$Z_1 = x^i \partial_j \Phi_1 - x^j \partial_i \Phi_1 + dx^i \wedge \iota(\partial_j)\, \Phi_1 - dx^j \wedge \iota(\partial_i)\, \Phi_1,$$

and similarly for W_0 and W_1 (with Ψ_0 and Ψ_1).

(d) $L_{B_i} \Phi$, for which the Cauchy data are

$$Z_0 = x^i \Psi_0 + t\partial_i \Phi_0 + \iota(\partial_i)\, \Phi_1,$$

$$Z_1 = x^i \Psi_1 + t\partial_i \Phi_1 + dx^i \wedge \Phi_0,$$

$$W_0 = x^i \left(-\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta} \delta \Psi_1\right) + t\partial_i \Psi_0 + \partial_i \Phi_0 + \iota(\partial_i)\, \Psi_1,$$

$$W_1 = x^i(-D\Phi_1 + (1-\beta)\, d\Psi_0) + t\partial_i \Psi_1 + \partial_i \Phi_1 + dx^i \wedge \Psi_0.$$

(e) $(L_H + (n-2k-2)/2)\, \Phi$, for which the Cauchy data are

$$Z_0 = t\Psi_0 + R\Phi_0 + \frac{n-2}{2} \Phi_0,$$

$$Z_1 = t\Psi_1 + R\Phi_1 + \frac{n-2}{2} \Phi_1,$$

$$W_0 = t \left(-\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta} \delta \Psi_1\right) + R\Psi_0 + \frac{n}{2} \Psi_0,$$

$$W_1 = t(-D\Phi_1 + (1-\beta)\, d\Psi_0) + R\Psi_1 + \frac{n}{2} \Psi_1.$$

(f) $(L_{I_n} + (n-2k-2)\, t)\, \Phi$, for which the Cauchy data are

$$Z_0 = (r^2 + t^2)\, \Psi_0 + 2tR\Phi_0 + (n-2)\, t\Phi_0 + 2\iota(R)\, \Phi_1,$$

$$Z_1 = (r^2 + t^2)\, \Psi_1 + 2tR\Phi_1 + (n-2)\, t\Phi_1 + 2\tilde{R} \wedge \Phi_0,$$

$$W_0 = (r^2 + t^2) \left(-\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta} \delta \Psi_1\right) + 2tR\Psi_0 + 2R\Phi_0 + nt\Psi_0$$
$$+ (n-2)\, \Phi_0 + 2\iota(R)\, \Psi_1,$$

$$W_1 = (r^2 + t^2)(-D\Phi_1 + (1-\beta)\, d\Psi_0) + 2tR\Psi_1 + 2R\Phi_1 + nt\Psi_1$$
$$+ (n-2)\, \Phi_1 + 2\tilde{R} \wedge \Psi_0.$$

Here $\tilde{R}$ is the one-form on $\mathbb{R}^{(n-1,0)}$ corresponding to R:

$$\tilde{R} = \sum_{i=1}^{n-1} x^i\, dx^i = \tfrac{1}{2} d(r^2).$$

(g) $(L_{I_i} - (n-2k-2)\, x^i)\, \Phi$, for which the Cauchy data are

$$\begin{aligned}
Z_0 &= -2x^i t \Psi_0 - 2x^i R \Phi_0 - 2dx^i \wedge \iota(R)\, \Phi_0 + (r^2 - t^2)\, \partial_i \Phi_0 \\
&\quad + 2R \wedge \iota(\partial_i)\, \Phi_0 - 2t\iota(\partial_i)\, \Phi_1 - (n-2)\, x^i \Phi_0, \\
Z_1 &= -2x^i t \Psi_1 - 2x^i R \Phi_1 - 2dx^i \wedge \iota(R)\, \Phi_1 + (r^2 - t^2)\, \partial_i \Phi_1 \\
&\quad + 2\tilde{R} \wedge \iota(\partial_i)\, \Phi_1 - 2t\, dx^i \wedge \Phi_0 - (n-2)\, x^i \Phi_1, \\
W_0 &= -2x^i t \left(-\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta}\, \delta \Psi_1 \right) - 2x^i R \Psi_0 - 2dx^i \wedge \iota(R)\, \Psi_0 \\
&\quad + (r^2 - t^2)\, \partial_i \Psi_0 - 2t \partial_i \Phi_0 + 2\tilde{R} \wedge \iota(\partial_i)\, \Psi_0 \\
&\quad - 2t\iota(\partial_i)\, \Psi_1 - 2\iota(\partial_i)\, \Phi_1 - n x^i \Psi_0, \\
W_1 &= -2x^i t(-D\Phi_1 + (1-\beta)\, d\Psi_0) - 2x^i R \Psi_1 - 2dx^i \wedge \iota(R)\, \Psi_1 \\
&\quad + (r^2 - t^2)\, \partial_i \Psi_1 - 2t \partial_i \Phi_1 + 2\tilde{R} \wedge \iota(\partial_i)\, \Psi_1 \\
&\quad - 2t\, dx^i \wedge \Psi_0 - 2dx^i \wedge \Phi_0 - n x^i \Psi_1.
\end{aligned}$$

Remark 2.1. $\partial/\partial t$ generates time translation; the ∂_i space translations; the A_{ij} space rotations; the B_i space-time "rotations", and H the *uniform dilations* $x \mapsto ax$, $a > 0$. I_n is the conjugation of $\partial/(\partial t)$ by the *inversion in the unit hyperboloid* Q: $(x, t) \mapsto (x, t)/(r^2 - t^2)$, and the I_i are the conjugations of the ∂_i by Q. The (Z_0, Z_1, W_0, W_1) are "formal vector fields" in the phase space for (2.3) representing the directions of conformal transforms of the point $(\Phi_0, \Phi_1, \Psi_0, \Psi_1)$.

3. Facts on Hamiltonian Systems in Finitely Many Degrees of Freedom

In Section 4, we reason heuristically in analogy with rigorous results on finite-dimensional Hamiltonian systems (reviewed in this section) to guess the quantities conserved by (2.3).

Let Ω be a closed 2-form on $\mathbb{R}^{2m}$ which is *nondegenerate*: given a vector $v \neq 0$ at a point x, there is another vector w with $\Omega_x(v, w) \neq 0$. A vector field T in $\mathbb{R}^{2m}$ is (locally) *Hamiltonian* if $L_T \Omega = 0$. To each Hamiltonian vector field T we associate the one-form $\iota(T)\, \Omega$, which is closed: $d\iota(T)\, \Omega = L_T \Omega = 0$ by (1.1) and the closure of Ω. By the Poincaré Lemma, $\iota(T)\, \Omega = d\xi_T$ for some function ξ_T with $\xi_T(0)=0$. Conversely, if f is a function, there is a unique vector field X_f with $df=\iota(X_f)\,\Omega$, i.e. $f=\xi_{X_f}+\text{constant}$, by the nondegeneracy of Ω. Thus if T is a Hamiltonian vector field and f a function,

$$Tf=\Omega(X_f,T).$$

Because the Lie derivative is a derivation of any naturally (i.e., pointwise) defined bilinear operation on tensor fields,

$$L_T\,d\xi_X=L_T\iota(X)\,\Omega=\iota([T,X])\,\Omega=d\xi_{[T,X]}$$

for Hamiltonian vector fields T and X; that is,

$$T\xi_X=\xi_{[T,X]}+\text{constant}. \tag{3.1}$$

A *Hamiltonian system* is the symplectic space $(\mathbb{R}^{2m},\Omega)$ together with a distinguished Hamiltonian vector field T. We denote by $\tilde{T}$ the vector field $T+\partial/\partial t$ on $\mathbb{R}^{2m}\times\mathbb{R}$ (the coordinate on $\mathbb{R}$ being t). *Solutions* of the system are integral curves of $\tilde{T}$ in $\mathbb{R}^{2m}\times\mathbb{R}$. A time-dependent vector field X on $\mathbb{R}^{2m}$, i.e., a vector field X on $\mathbb{R}^{2m}\times\mathbb{R}$ with no $\partial/\partial t$ component, Hamiltonian at each fixed time, is a *symmetry* of the system if $[X,\tilde{T}]=0$ (that is, if each transformation in the flow of X permutes the solutions of the system.)

Since

$$[X,\tilde{T}]=-\dot{X}+[X,T]$$

(the circumscribed dot denoting $L_{\partial/\partial t}$), the condition that X is a symmetry implies $\dot{X}=[X,T]$, so that

$$\dot{\xi}_X=\xi_{[X,T]}=-T\xi_X+\alpha(t),$$

where α is a C^∞ function of t alone. Thus $\tilde{T}\xi_X=\alpha(t)$, or $\tilde{T}(\xi_X-A(t))=0$, where $\alpha=dA/dt$. This means that the quantity $\xi_X-A(t)$ is *conserved* by solutions of the Hamiltonian system. If X is time-independent, α is constant and $A(t)$ may be chosen to be αt.

As an example, consider the ODE $\ddot{x}+|x|^2=0$ on a function $x\colon\mathbb{R}\to\mathbb{R}^2$. The *phase space* for this equation is $\mathbb{R}^4$ (whose coordinates will be denoted $q_1,\ q_2,\ p_1,\ p_2$) with the symplectic form $\Omega=dq_1\wedge dp_1+dq_2\wedge dp_2$. Interpreting p_i as the time derivative of q_i, the ODE says that

$$(q,p)^{\cdot}=(p,-|q|^2\,q). \tag{3.2}$$

Thus we set $T=p_1(\partial/\partial q_1)+p_2(\partial/\partial q_2)-|q|^2\,q_1(\partial/\partial p_1)-|q|^2\,q_2(\partial/\partial p_2)$; T is easily seen to be Hamiltonian. The vector field $X=-q_2(\partial/\partial q_1)+q_1(\partial/\partial q_2)-p_2(\partial/\partial p_1)+p_1(\partial/\partial p_2)$ is also Hamiltonian, with $[X,T]=0$. T and X are thus symmetries of the Hamiltonian system $(\mathbb{R}^4, \Omega, T)$, and the corresponding conserved quantities may be calculated as follows: if $Y = Q_1(\partial/\partial q_1) + Q_2(\partial/\partial q_2) + P_1(\partial/\partial p_1) + P_2(\partial/\partial p_2)$ is a typical vector field in $\mathbb{R}^4$,

$$\Omega(T, Y) = p \cdot P + |q|^2\, q \cdot Q,$$

where the dot products are in $\mathbb{R}^2$. This is the *linearization* (derivative at (q, p) applied to (Q, P)) of the *energy*

$$\xi_T = \tfrac{1}{2}\,|p|^2 + \tfrac{1}{4}\,|q|^4.$$

It is easily checked that $\dot{\xi}_T = 0$ for a solution (q, p) to (3.2). As for X, $\Omega(X, Y) = -q_2 P_1 + q_1 P_2 + p_2 Q_1 - p_1 Q_2$, which is the linearization of the *angular momentum* $\xi_X = q_1 p_2 - q_2 p_1$; checking, we find $\dot{\xi}_X = 0$ for solutions (q, p).

4. Symplectic Structure of the Equations and Heuristic Determination of the Conserved Quantities

With a correct choice of (formal) symplectic structure, the type of calculation immediately above may be carried out formally to "guess" conserved quantities of (2.3). The existence and conservation of these quantities (rigorously treated in Section 5) will always depend on some decay assumptions at infinity for solutions and their space derivatives. We denote by $\langle \omega, \eta \rangle$ the (time-dependent) L^2 inner product of time-dependent forms over $\mathbb{R}^{n-1}$:

$$\langle \omega, \eta \rangle = \int_{\mathbb{R}^{n-1}} \omega \cdot \eta.$$

Here we write each of the inner products g^k in $\mathbb{R}^{n-1}$ as a dot product, since it is the same as the dot product obtained when k-forms are written as vectors with $(n-1)!/k!\,(n-1-k)!$ components in the standard basis. All forms are assumed smooth; for example, "$\omega \in L^2$" means that all components of ω are C^∞ in x and t and L^2 in x. We assume that $k \neq (n \pm 2)/2$.

The *symplectic structure* for (2.3) is the alternating bilinear form

$$\begin{aligned}\Omega((Z, W), (X, Y)) = {} & \langle Z_1, Y_1 \rangle - \langle X_1, W_1 \rangle - \beta \langle Z_0, Y_0 \rangle + \beta \langle X_0, W_0 \rangle \\ & - (1-\beta)\, \langle Z_1, dX_0 \rangle + (1-\beta)\, \langle \delta X_1, Z_0 \rangle, \end{aligned} \tag{4.1}$$

where $Z = (Z_0, Z_1)$, Z_0 being a $(k-1)$-form on $\mathbb{R}^{n-1}$, and Z_1 a k-form; etc. The quantity on the right is defined, for example, when Z_0, Z_1, W_0, W_1, dX_0, X_1, δX_1, Y_0, Y_1 are L^2. Ω is (formally) nondegenerate: if $\beta > 0$

($k \notin [(n-2)/2, (n+2)/2]$) and (Z, W) is given, we set $X_0 = W_0$, $X_1 = -W_1$, $Y_0 = -Z_0 - ((1-\beta)/\beta)\,\delta W_1$, $Y_1 = Z_1 + (1-\beta)\,dW_0$ to get $\Omega((Z, W), (X, Y)) = \langle Z_1, Z_1\rangle + \langle W_1, W_1\rangle + \beta\langle Z_0, Z_0\rangle + \beta\langle W_0, W_0\rangle$. If $\beta < 0$ ($k = n/2$ for even n or $k = (n \pm 1)/2$ for odd n), we set $X_0 = -W_0$, $X_1 = -W_1$, $Y_0 = Z_0 - ((1-\beta)/\beta)\,\delta W_1$, $Y_1 = Z_1 - (1-\beta)\,dW_0$ to get $\Omega((Z, W), (X, Y)) = \langle Z_1, Z_1\rangle + \langle W_1, W_1\rangle - \beta\langle Z_0, Z_0\rangle - \beta\langle W_0, W_0\rangle$. Ω is formally closed because its "coefficients" are constant: the right side of (4.1) makes no mention of the point (Φ, Ψ) at which the vectors (Z, W), (X, Y) are tangent.

Remark 4.1. When $k = 0$, all quantities with the subscript zero vanish, and Ω becomes the standard symplectic structure for the wave equation, $\Omega((Z, W), (X, Y)) = \langle Z, Y\rangle - \langle X, W\rangle$; Z, W, X, and Y being functions on $\mathbb{R}^{n-1}$. A more subtle point is that (4.1) also agrees with the symplectic structure for the Maxwell equations (the case $k = (n-2)/2$; this is discussed in Remark 7.1.

Remark 4.2. The symplectic structure (4.1) was chosen as follows. We wish the system (2.2) to have a conserved *energy* of the form

$$\begin{aligned} &a_0\langle \Psi_0, \Psi_0\rangle + a_1\langle \Psi_1, \Psi_1\rangle + b_0\langle d\Phi_0, d\Phi_0\rangle + c_0\langle \delta\Phi_0, \delta\Phi_0\rangle \\ &\quad + b_1\langle d\Phi_1, d\Phi_1\rangle + c_1\langle \delta\Phi_1, \delta\Phi_1\rangle, \end{aligned} \tag{4.2}$$

the a_i, b_i, and c_i constants, in analogy with the energy $\frac{1}{2}\langle \psi, \psi\rangle + \frac{1}{2}\langle d\phi, d\phi\rangle$ of the wave equation $\square_0 \phi = 0$ (here $\psi = \dot{\phi}$). Formally differentiating (4.2) with respect to t, we get

$$\begin{aligned} &2a_0\left\langle \Psi_0, -\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta}\,\delta\Psi_1\right\rangle + 2a_1\langle \Psi_1, -D\Phi_1 + (1-\beta)\,d\Psi_0\rangle \\ &\quad + 2b_0\langle d\Phi_0, d\Psi_0\rangle + 2c_0\langle \delta\Phi_0, \delta\Psi_0\rangle \\ &\quad + 2b_1\langle d\Phi_1, d\Psi_1\rangle + 2c_1\langle \delta\Phi_1, \delta\Psi_1\rangle \\ &= \left(-\frac{2a_0}{\beta} + 2b_0\right)\langle d\Phi_0, d\Psi_0\rangle + (-2a_0 + 2c_0)\,\langle \delta\Phi_0, \delta\Psi_0\rangle \\ &\quad + (-2a_1 + 2b_1)\,\langle d\Phi_1, d\Psi_1\rangle + (-2a_1\beta + 2c_1)\,\langle \delta\Phi_1, \delta\Psi_1\rangle \\ &\quad + (1-\beta)\left(\frac{2a_0}{\beta} + 2a_1\right)\langle d\Psi_0, \Psi_1\rangle. \end{aligned}$$

For this to be identically 0 we need

$$a_0 = -\beta a_1, \qquad b_0 = -a_1, \qquad c_0 = -\beta a_1,$$
$$b_1 = a_1, \qquad c_1 = \beta a_1.$$

Thus the energy is determined up to a constant multiple, and may be taken to be

$$-\frac{\beta}{2}\langle\Psi_0,\Psi_0\rangle+\frac{1}{2}\langle\Psi_1,\Psi_1\rangle-\frac{1}{2}\langle d\Phi_0,d\Phi_0\rangle-\frac{\beta}{2}\langle\delta\Phi_0,\delta\Phi_0\rangle$$
$$+\frac{1}{2}\langle d\Phi_1,d\Phi_1\rangle+\frac{\beta}{2}\langle\delta\Phi_1,\delta\Phi_1\rangle. \tag{4.3}$$

(This agrees with the wave equation energy when $k=0$.) The linearization of the energy at (Φ, Ψ) applied to a vector (X, Y) is

$$-\beta\langle\Psi_0,Y_0\rangle+\langle\Psi_1,Y_1\rangle-\langle d\Phi_0,dX_0\rangle-\beta\langle\delta\Phi_0,\delta X_0\rangle$$
$$+\langle d\Phi_1,dX_1\rangle+\beta\langle\delta\Phi_1,\delta X_1\rangle. \tag{4.4}$$

This should be $\Omega((Z, W), (X, Y))$, where (Z, W) is the vector field of time translation: $Z_0=\Psi_0$, $Z_1=\Psi_1$, $W_0=-(1/\beta)\,D\Phi_0+((1-\beta)/\beta)\,\delta\Psi_1$, $W_1=-D\Phi_1+(1-\beta)\,d\Psi_0$. (4.1) is the simplest choice for which this is so, and which agrees with the usual wave equation symplectic structure when $k=0$.

We now calculate the Hamiltonian functions associated to the formal vector fields (Z_0, Z_1, W_0, W_1) of (a)–(g) at the end of Section 2. The method is to calculate $\Omega((Z, W), (X, Y))$ for a typical formal vector field (X, Y), and then "unlinearize". We denote the linearization of, for example, $\frac{1}{2}\langle\Psi_1,\Psi_1\rangle$, by $\operatorname{Lin}\{\frac{1}{2}\langle\Psi_1,\Psi_1\rangle\}$:

$$\langle\Psi_1,Y_1\rangle=\operatorname{Lin}\{\tfrac{1}{2}\langle\Psi_1,\Psi_1\rangle\}.$$

Note that the fact that $\Omega((Z, W), (X, Y))$ is the linearization of some quantity formally shows that (Z, W) is Hamiltonian (see Section 3). The Hamiltonian character of the (Z, W) associated to each conformal T is formally equivalent to the *conformal invariance* of Ω.

Before beginning the calculations, we collect some identities useful here and in Section 5. Let ω and η be p-forms, and ξ a $(p-1)$-form on $\mathbb{R}^{n-1}$. Then

$$\partial_i(\omega\cdot\eta)=\omega\cdot\partial_i\eta+\partial_i\omega\cdot\eta,$$
$$\sum_{i=1}^{n-1}\partial_i(\iota(\partial_i)\,\omega\cdot\xi)=\omega\cdot d\xi-\delta\omega\cdot\xi.$$

Thus by Stokes' Theorem, setting $|\omega|^2=\omega\cdot\omega$,

$$\langle\omega,\partial_i\eta\rangle=-\langle\partial_i\omega,\eta\rangle \qquad \text{if}\quad |\omega|\,|\eta| \text{ vanishes at } \infty, \tag{4.5}$$
$$\langle\omega,d\xi\rangle=\langle\delta\omega,\xi\rangle \qquad \text{if}\quad |\omega|\,|\xi| \text{ vanishes at } \infty, \tag{4.6}$$

$$\langle\omega, R\eta\rangle = -\langle R\omega, \eta\rangle - (n-1)\langle\omega, \eta\rangle \qquad \text{if} \quad r|\omega||\eta| \text{ vanishes at } \infty, \tag{4.7}$$

$$\langle\omega, \partial_i\omega\rangle = 0 \qquad \text{if} \quad |\omega|^2 \text{ vanishes at } \infty, \tag{4.8}$$

$$\langle\omega, R\omega\rangle = -\frac{n-1}{2}\langle\omega, \omega\rangle \qquad \text{if} \quad r|\omega|^2 \text{ vanishes at } \infty. \tag{4.9}$$

If f is a function and ω a p-form,

$$dR\omega = Rd\omega + d\omega, \tag{4.10}$$

$$\delta R\omega = R\delta\omega + \delta\omega, \tag{4.11}$$

$$d(f\omega) = fd\omega + df \wedge \omega, \tag{4.12}$$

$$\delta(f\omega) = f\delta\omega - \iota(\widetilde{df})\,\omega, \tag{4.13}$$

$$L_R\omega = R\omega + p\omega. \tag{4.14}$$

For (Z, W) as in (a),

$$\begin{aligned}
\Omega((Z, W), (X, Y)) &= \langle\Psi_1, Y_1\rangle - \langle X_1, -D\Phi_1 + (1-\beta)\, d\Psi_0\rangle \\
&\quad - \beta\langle\Psi_0, Y_0\rangle + \beta\left\langle X_0, -\frac{1}{\beta}D\Phi_0 + \frac{1-\beta}{\beta}\delta\Psi_1\right\rangle \\
&\quad - (1-\beta)\langle\Psi_1, dX_0\rangle + (1-\beta)\langle\delta X_1, \Psi_0\rangle \\
&= \mathrm{Lin}\left\{\frac{1}{2}\langle\Psi_1, \Psi_1\rangle - \frac{\beta}{2}\langle\Psi_0, \Psi_0\rangle\right\} \\
&\quad + \langle dX_1, d\Phi_1\rangle + \beta\langle\delta X_1, \delta\Phi_1\rangle \\
&\quad - \langle dX_0, d\Phi_0\rangle - \beta\langle\delta X_0, \delta\Phi_0\rangle \\
&= \mathrm{Lin}\left\{\frac{1}{2}\langle\Psi_1, \Psi_1\rangle - \frac{\beta}{2}\langle\Psi_0, \Psi_0\rangle\right. \\
&\quad + \frac{1}{2}\langle d\Phi_1, d\Phi_1\rangle + \frac{\beta}{2}\langle\delta\Phi_1, \delta\Phi_1\rangle \\
&\quad \left. - \frac{1}{2}\langle d\Phi_0, d\Phi_0\rangle - \frac{\beta}{2}\langle\delta\Phi_0, \delta\Phi_0\rangle\right\}.
\end{aligned}$$

Thus we get the energy (4.3), defined when Ψ_0, Ψ_1, Φ_0', Φ_1' are L^2 at each fixed time (where Φ_0' denotes the complex of all first x-derivatives of Φ_0). The *energy density* is the function

$$\begin{aligned}
E &= \frac{1}{2}\Psi_1 \cdot \Psi_1 - \frac{\beta}{2}\Psi_0 \cdot \Psi_0 + \frac{1}{2}d\Phi_1 \cdot d\Phi_1 + \frac{\beta}{2}\delta\Phi_1 \cdot \delta\Phi_1 \\
&\quad - \frac{1}{2}d\Phi_0 \cdot d\Phi_0 - \frac{\beta}{2}\delta\Phi_0 \cdot \delta\Phi_0.
\end{aligned} \tag{4.15a}$$

The energy density for the wave equation $\psi = \dot{\phi}$, $\dot{\psi} = -\Delta\phi$ is thus $E_0 = \frac{1}{2}\psi^2 + \frac{1}{2}d\phi \cdot d\phi$.

By similar calculations, the quantities associated to the vector fields (Z, W) in (b) are the integrals over $\mathbb{R}^{n-1}$ of the *linear momentum densities*

$$p_i = -\beta\partial_i\Phi_0 \cdot \Psi_0 + \partial_i\Phi_1 \cdot \Psi_1 - (1-\beta)\,\partial_i\Phi_1 \cdot d\Phi_0. \tag{4.15b}$$

For the integrals to exist, we again require Ψ_0, Ψ_1, Φ_0', and Φ_1' to be L^2 at each fixed time. For the wave equation, this reduces to $(\partial_i\phi)\,\psi$.

Corresponding to the vector fields in (c), we get the *angular momentum dentities*

$$\begin{aligned} x^i p_j - x^j p_i &- \beta\iota(\partial_j)\,\Phi_0 \cdot \iota(\partial_i)\,\Psi_0 + \beta\iota(\partial_i)\,\Phi_0 \cdot \iota(\partial_j)\,\Psi_0 \\ &+ \iota(\partial_j)\,\Phi_1 \cdot \iota(\partial_i)\,\Psi_1 - \iota(\partial_i)\,\Phi_1 \cdot \iota(\partial_j)\,\Psi_1 \\ &- (1-\beta)\,\iota(\partial_j)\,\Phi_1 \cdot \iota(\partial_i)\,d\Phi_0 \\ &+ (1-\beta)\,\iota(\partial_i)\,\Phi_1 \cdot \iota(\partial_j)\,d\Phi_0, \end{aligned} \tag{4.15c}$$

whose integrals exist when $\sqrt{r}\Psi_0$, $\sqrt{r}\Psi_1$, $\sqrt{r}\Phi_0'$, $\sqrt{r}\Phi_1'$, and Φ_1 are L^2 at each fixed time. For the wave equation, this reduces to $x^i p_j - x^j p_i$, the integral of which exists when $\sqrt{r}\psi$ and $\sqrt{r}d\phi$ are L^2.

For (d), we get the *space-time angular momentum densities*

$$\begin{aligned} x^i E + tp_i &- \beta\iota(\partial_i)\,\Phi_1 \cdot \Psi_0 + \Phi_0 \cdot \iota(\partial_i)\,\Psi_1 \\ &- \beta\Phi_1 \cdot d\iota(\partial_i)\,\Phi_1 - \Phi_0 \cdot \iota(\partial_i)\,d\Phi_0, \end{aligned} \tag{4.15d}$$

whose integrals exist if $\sqrt{r}\Psi_0$, $\sqrt{r}\Psi_1$, $\sqrt{r}\Phi_0'$, $\sqrt{r}\Phi_1'$, Φ_0, and Φ_1 are L^2 at each fixed time. For the wave equation, this reduces to $x^i E + tp_j$, the integral of which exists if $\sqrt{r}\psi$ and $\sqrt{r}d\phi$ are L^2. For (e), we get the *dilational density*

$$\begin{aligned} tE + \sum_{i=1}^{n-1} x^i p_i &- \frac{n-2}{2}\beta\Phi_0 \cdot \Psi_0 + \frac{n-2}{2}\Phi_1 \cdot \Psi_1 \\ &+ \frac{n-2}{2}(1-\beta)\,\Phi_1 \cdot d\Phi_0. \end{aligned} \tag{4.15e}$$

The integral over $\mathbb{R}^{n-1}$ is defined if $\sqrt{r}\Psi_0$, $\sqrt{r}\Psi_1$, $\sqrt{r}\Phi_0'$, $\sqrt{r}\Phi_1'$, Φ_0, and Φ_1 are L^2 at each fixed time. For the wave equation, this reduces to $tE + \sum_{i=1}^{n-1} x^i p_i + ((n-2)/2)\,\phi\psi$.

We include the calculations for the nth *inversional density*, corresponding to (f), in full detail. For (Z, W) as in (f),

$$\begin{aligned}
&\Omega((Z, W), (X, Y)) \\
&\quad = \langle (r^2 + t^2)\, \Psi_1 + 2tR\Phi_1 + (n-2)\, t\Phi_1 + 2\tilde{R} \wedge \Phi_0, Y_1\rangle \\
&\qquad - \langle X_1, (r^2 + t^2)(-D\Phi_1 + (1-\beta)\, d\Psi_0) + 2tR\Psi_1 \\
&\qquad + 2R\Phi_1 + nt\Psi_1 + (n-2)\, \Phi_1 + 2\tilde{R} \wedge \Psi_0\rangle \\
&\qquad - \beta\langle (r^2 + t^2)\, \Psi_0 + 2tR\Phi_0 + (n-2)\, t\Phi_0 + 2\iota(R)\, \Phi_1, Y_0\rangle \\
&\qquad + \beta\left\langle X_0, (r^2 + t^2)\left(-\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta}\, \delta\Psi_1\right) + 2tR\Psi_0 \right. \\
&\qquad \left. + 2R\Phi_0 + nt\Psi_0 + (n-2)\, \Phi_0 + 2\iota(R)\, \Psi_1 \right\rangle \\
&\qquad - (1-\beta)\, \langle (r^2 + t^2)\, \Psi_1 + 2tR\Phi_1 + (n-2)\, t\Phi_1 \\
&\qquad + 2\tilde{R} \wedge \Phi_0, dX_0\rangle \\
&\qquad + (1-\beta)\, \langle \delta X_1, (r^2 + t^2)\, \Psi_0 \\
&\qquad + 2tR\Phi_0 + (n-2)\, t\Phi_0 + 2\iota(R)\, \Phi_1\rangle.
\end{aligned}$$

The terms involving Ψ_1 and Y_1 reduce to

$$\mathrm{Lin}\{\tfrac{1}{2}t^2\langle \Psi_1, \Psi_1\rangle + \tfrac{1}{2}\langle r\Psi_1, r\Psi_1\rangle\}, \tag{4.16}$$

and those involving Ψ_0 and Y_0 to

$$\mathrm{Lin}\left\{-\frac{\beta}{2} t^2\langle \Psi_0, \Psi_0\rangle - \frac{\beta}{2}\langle r\Psi_0, r\Psi_0\rangle\right\}. \tag{4.17}$$

Terms involving Φ_1 and Y_1 or Ψ_1 and X_1 are

$$2t\langle R\Phi_1, Y_1\rangle + (n-2)\, t\langle \Phi_1, Y_1\rangle - 2t\langle X_1, R\Psi_1\rangle - nt\langle X_1, \Psi_1\rangle,$$

or, by (4.7),

$$\mathrm{Lin}\{2t\langle R\Phi_1, \Psi_1\rangle + (n-2)\, t\langle \Phi_1, \Psi_1\rangle\}. \tag{4.18}$$

Terms involving Φ_0 and Y_1 or Ψ_1 and X_0 are

$$\begin{aligned}
&2\langle \tilde{R} \wedge \Phi_0, Y_1\rangle + (1-\beta)\, t^2\langle X_0, \delta\Psi_1\rangle + (1-\beta)\, \langle X_0, r^2\delta\Psi_1\rangle \\
&\qquad + 2\beta\langle X_0, \iota(R)\, \Psi_1\rangle - (1-\beta)\, t^2\langle \Psi_1, dX_0\rangle - (1-\beta)\, \langle r^2\Psi_1, dX_0\rangle,
\end{aligned}$$

the second and fifth of which cancel. Using (4.12), this becomes

$$\mathrm{Lin}\{2\langle \tilde{R} \wedge \Phi_0, \Psi_1\rangle\}. \tag{4.19}$$

Terms involving Φ_1 and X_1 are

$$t^2\langle X_1, D\Phi_1\rangle + \langle r^2 X_1, D\Phi_1\rangle - 2\langle X_1, R\Phi_1\rangle - (n-2)\langle X_1, \Phi_1\rangle$$
$$+ 2(1-\beta)\langle \delta X_1, \iota(R)\Phi_1\rangle.$$

Using (4.13), this becomes

$$\begin{aligned}\operatorname{Lin}\Big\{&\frac{1}{2}t^2\langle d\Phi_1, d\Phi_1\rangle + \frac{\beta}{2}t^2\langle \delta\Phi_1, \delta\Phi_1\rangle \\ &+ \frac{1}{2}\langle r d\Phi_1, r d\Phi_1\rangle + \frac{\beta}{2}\langle r\delta\Phi_1, r\delta\Phi_1\rangle\Big\} \\ &+ 2\langle \tilde{R}\wedge X_1, d\Phi_1 > - 2\beta\langle \iota(R) X_1, \delta\Phi_1\rangle - 2\langle X_1, R\Phi_1\rangle \\ &- (n-2)\langle X_1, \Phi_1\rangle + 2(1-\beta)\langle X_1, d\iota(R)\Phi_1\rangle,\end{aligned}$$

which equals

$$\begin{aligned}\operatorname{Lin}\Big\{&\frac{1}{2}t^2\langle d\Phi_1, d\Phi_1\rangle + \frac{\beta}{2}t^2\langle \delta\Phi_1, \delta\Phi_1\rangle + \frac{1}{2}\langle r d\Phi_1, r d\Phi_1\rangle \\ &+ \frac{\beta}{2}\langle r\delta\Phi_1, r\delta\Phi_1\rangle - 2\beta\langle \Phi_1, \tilde{R}\wedge\delta\Phi_1\rangle - \frac{n-2k-2}{2}\langle \Phi_1, \Phi_1\rangle\Big\},\end{aligned} \tag{4.20}$$

using (4.14) and (1.1). Terms involving Φ_1 and Y_0 or Ψ_0 and X_1 are

$$-(1-\beta)t^2\langle X_1, d\Psi_0\rangle - (1-\beta)\langle X_1, r^2 d\Psi_0\rangle - 2\langle X_1, \tilde{R}\wedge\Psi_0\rangle$$
$$- 2\beta\langle \iota(R)\Phi_1, Y_0\rangle + (1-\beta)t^2\langle \delta X_1, \Psi_0\rangle + (1-\beta)\langle \delta X_1, r^2\Psi_0\rangle,$$

the first and fifth of which cancel, and the rest of which reduces to

$$\operatorname{Lin}\{-2\beta\langle \Phi_1, \tilde{R}\wedge\Psi_0\rangle\}. \tag{4.21}$$

Terms involving Φ_0 and Y_0 or Ψ_0 and X_0 are

$$-2\beta t\langle R\Phi_0, Y_0\rangle - (n-2)\beta t\langle \Phi_0, Y_0\rangle + 2\beta t\langle X_0, R\Psi_0\rangle + n\beta t\langle X_0, \Psi_0\rangle$$

which equals

$$\operatorname{Lin}\{-2\beta t\langle R\Phi_0, \Psi_0\rangle - (n-2)\beta t\langle \Phi_0, \Psi_0\rangle\} \tag{4.22}$$

by (4.7). Terms involving Φ_0 and X_0 are

$$-t^2\langle X_0, D\Phi_0\rangle - \langle X_0, r^2 D\Phi_0\rangle + 2\beta\langle X_0, R\Phi_0\rangle + (n-2)\beta\langle X_0, \Phi_0\rangle$$
$$- 2(1-\beta)\langle \tilde{R}\wedge\Phi_0, dX_0\rangle.$$

By (4.9) and an argument similar to that giving (4.11), this is

$$\operatorname{Lin}\left\{-\frac{1}{2}t^2\langle d\Phi_0, d\Phi_0\rangle-\frac{\beta}{2}t^2\langle\delta\Phi_0,\delta\Phi_0\rangle-\frac{1}{2}\langle rd\Phi_0, rd\Phi_0\rangle\right.$$
$$\left.-\frac{\beta}{2}\langle r\delta\Phi_0, r\delta\Phi_0\rangle-2\langle\Phi_0,\iota(R)\,d\Phi_0\rangle-\frac{n-2k+2}{2}\beta\langle\Phi_0,\Phi_0\rangle\right\}. \tag{4.23}$$

Finally, the terms involving Φ_1 and X_0 or Φ_0 and X_1 give

$$-2(1-\beta)\,t\langle R\Phi_1, dX_0\rangle-(n-2)(1-\beta)\,t\langle\Phi_1, dX_0\rangle$$
$$+2(1-\beta)\,t\langle\delta X_1, R\Phi_0\rangle+(n-2)(1-\beta)\,t\langle\delta X_1,\Phi_0\rangle.$$

By (4.10), this is

$$\operatorname{Lin}\{-2(1-\beta)\,t\langle R\Phi_1, d\Phi_0\rangle-(n-2)(1-\beta)\,t\langle\Phi_1, d\Phi_0\rangle\} \tag{4.24}$$

Adding (4.16)–(4.24), the quantity corresponding to (f) is the integral over $\mathbb{R}^{n-1}$ of

$$(r^2+t^2)\,E+2t\sum_{i=1}^{n-1}x^ip_i-\tfrac{1}{2}(n-2k+2)\,\beta\Phi_0\cdot\Phi_0$$
$$-\tfrac{1}{2}(n-2k-2)\,\Phi_1\cdot\Phi_1-(n-2)\,\beta t\Phi_0\cdot\Psi_0+(n-2)\,t\Phi_1\cdot\Psi_1$$
$$-2\beta\Phi_1\cdot(\tilde{R}\wedge\Psi_0)+2\Phi_0\cdot\iota(R)\,\Psi_1-2\Phi_0\cdot\iota(R)\,d\Phi_0$$
$$-2\beta\Phi_1\cdot d\iota(R)\,\Phi_1-(n-2)(1-\beta)\,t\Phi_1\cdot d\Phi_0. \tag{4.15f}$$

This integral exists if $r\Psi_0$, $r\Psi_1$, $r\Phi_0'$, $r\Phi_1'$, Φ_0, and Φ_1 are L^2 at each fixed time. For the wave equation, (4.15f) reduces to

$$(r^2+t^2)\,E+2t\sum x^ip_i-\tfrac{1}{2}(n-2)\,\phi^2+(n-2)\,t\phi\psi.$$

Remark 4.3. Up to this point, we have not used the fact that $\beta=(n-2k-2)/(n-2k+2)$; the vector fields (Z, W) of (a)–(f) are thus formally Hamiltonian regardless of the value of β. This is not the case for (g), as we shall see below. In Section 5, we show that the integrals of (4.15a)–(4.15e) are conserved (with decay assumptions on Φ and Ψ) for any value of β; but that the integrals of (4.15g) below *and* (4.15f) are conserved only for $\beta=(n-2k-2)/(n-2k+2)$.

The density corresponding to the (Z, W) of (g) is

$$
\begin{aligned}
&-2tx^iE - 2x^i \sum_{j=1}^{n-1} x^j p_j + (r^2 - t^2)\, p_i \\
&\quad + 2t\Phi_0 \cdot \iota(\partial_i)\, d\Phi_0 + 2\beta t\Phi_1 \cdot d\iota(\partial_i)\, \Phi_1 - 2\beta\Phi_0 \cdot (dx^i \wedge \iota(R)\, \Psi_0) \\
&\quad + 2\beta\Phi_0 \cdot (\tilde{R} \wedge \iota(\partial_i)\, \Psi_0) + \beta(n-2)\, \Phi_0 \cdot x^i \Psi_0 + 2\Phi_1 \cdot (dx^i \wedge \iota(R)\, \Psi_1) \\
&\quad - 2\Phi_1 \cdot (\tilde{R} \wedge \iota(\partial_i)\, \Psi_1) - (n-2)\, \Phi_1 \cdot x^i \Psi_1 + 2\beta t\Phi_1 \cdot (dx^i \wedge \Psi_0) \\
&\quad - 2t\Phi_0 \cdot \iota(\partial_i)\, \Psi_1 + 2(1-\beta)\, \Phi_1 \cdot (\tilde{R} \wedge \iota(\partial_i)\, d\Phi_0) \\
&\quad - 2(1-\beta)\, \Phi_1 \cdot (dx^i \wedge \iota(R)\, d\Phi_0) \\
&\quad - 2\beta\Phi_0 \cdot \iota(\partial_i)\, \Phi_1 + (n-2)(1-\beta)\, \Phi_1 \cdot x^i d\Phi_0.
\end{aligned} \tag{4.15g}
$$

The integral over $\mathbb{R}^{n-1}$ exists if $r\Psi_0$, $r\Psi_1$, $r\Phi_0'$, $r\Phi_1'$, Φ_0, and Φ_1 are L^2 at each fixed time. For the wave equation, (4.15g) reduces to $-2tx^iE - 2x^i \sum x^j p_j + (r^2 - t^2)\, p_i - (n-2)\, x^i \phi\psi$.

We indicate briefly why the value $\beta = (n-2k-2)/(n-2k+2)$ is essential in the derivation of (4.15g). The terms in $\Omega((Z, W), (X, Y))$ involving Φ_0 and X_1 or Φ_1 and X_0 must come from the second, fourth, fifth, and sixth terms on the right in (4.1). From the second we get $2\langle X_1, dx^i \wedge \Phi_0\rangle$, from the fourth $-2\beta\langle X_0, \iota(\partial_i)\, \Phi_1\rangle$, from the fifth

$$
\begin{aligned}
&2(1-\beta)\, \langle x^i R\Phi_1, dX_0\rangle + 2(1-\beta)\, \langle dx^i \wedge \iota(R)\, \Phi_1, dX_0\rangle \\
&\quad + (1-\beta)\, t^2 \langle \partial_i \Phi_1, dX_0\rangle - (1-\beta)\, \langle r^2 \partial_i \Phi_1, dX_0\rangle \\
&\quad - 2(1-\beta)\, \langle \tilde{R} \wedge \iota(\partial_i)\, \Phi_1, dX_0\rangle + (n-2)(1-\beta)\, \langle x^i \Phi_1, dX_0\rangle,
\end{aligned}
$$

and from the sixth

$$
\begin{aligned}
&-2(1-\beta)\, \langle \delta X_1, x^i R\Phi_0\rangle - 2(1-\beta)\, \langle \delta X_1, dx^i \wedge \iota(R)\, \Phi_0\rangle \\
&\quad - (1-\beta)\, t^2 \langle \delta X_1, \partial_i \Phi_0\rangle + (1-\beta)\, \langle \delta X_1, r^2 \partial_i \Phi_0\rangle \\
&\quad + 2(1-\beta)\, \langle \delta X_1, \tilde{R} \wedge \iota(\partial_i)\, \Phi_0\rangle - (n-2)(1-\beta)\, \langle \delta X_1, x^i \Phi_0\rangle.
\end{aligned}
$$

Adding and arguing as above without assigning any particular value to β, we get

$$
\begin{aligned}
&\mathrm{Lin}\{2(1-\beta)\, \langle R\Phi_1, x^i d\Phi_0\rangle - (1-\beta)\, \langle r\partial_i \Phi_1, r d\Phi_0\rangle \\
&\quad + 2(1-\beta)\, \langle \Phi_1, \tilde{R} \wedge \iota(\partial_i)\, d\Phi_0\rangle + (1-\beta)\, t^2 \langle \partial_i \Phi_1, d\Phi_0\rangle \\
&\quad - 2(1-\beta)\, \langle \Phi_1, dx^i \wedge \iota(R)\, d\Phi_0\rangle + (n-2)(1-\beta)\, \langle \Phi_1, x^i d\Phi_0\rangle\} \\
&\quad - 2\beta\langle X_0, \iota(\partial_i)\, \Phi_1\rangle + [(1-\beta)(-n+2k) + 2]\, \langle \Phi_0, \iota(\partial_i)\, X_1\rangle.
\end{aligned}
$$

For this to be the linearization of some quantity, we need $-2\beta = (1-\beta)(-n+2k) + 2$, or $\beta = (n-2k-2)/(n-2k+2)$.

5. Rigorous Justification of the Conservation Laws

We show that the integrals over $\mathbb{R}^{n-1}$ of the densities (4.15a)–(4.15g) are conserved by the equations (2.3) by showing that their time derivatives are exact divergences of (time-dependent) vector fields which vanish at ∞. The calculations are carried out in detail only for (4.15a) and (4.15f); the others are entirely similar. We say that functions f and g are *equivalent*, $f \cong g$, if $f-g$ is the divergence of a vector field vanishing at ∞. Because of the way (4.5)–(4.9) are derived, each is an equivalence: for example, (4.5) may be replaced by $\omega \cdot \partial_i \eta \cong -\partial_i \omega \cdot \eta$.

Using (2.3) to calculate the time derivative of (4.15a),

$$\dot{E} = \Psi_1 \cdot \{-D\Phi_1 + (1-\beta)\, d\Psi_0\} - \beta \Psi_0 \cdot \left\{-\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta} \delta \Psi_1\right\}$$
$$+ d\Phi_1 \cdot d\Psi_1 + \beta \delta \Phi_1 \cdot \delta \Psi_1 - d\Phi_0 \cdot d\Psi_0 - \beta \delta \Phi_0 \cdot \delta \Psi_0.$$

This is equivalent to zero provided $|\Psi_1||d\Phi_1|$, $|\Psi_1||\delta\Phi_1|$, $|\Psi_0||d\Phi_0|$, $|\Psi_0||\delta\Phi_0|$, and $|\Psi_0||\Psi_1|$ vanish at ∞. Thus the assumptions Ψ_0, Ψ_1, Φ_0', $\Phi_1' \in L^2$ are enough to guarantee that $\int E$ is conserved. (This and all integrals are over $\mathbb{R}^{n-1}$.)

Similarly, $\int p_i$ is conserved with the same decay assumptions; the integrals of the angular momentum densities (4.15c) if $\sqrt{r}\Psi_0$, $\sqrt{r}\Psi_1$, $\sqrt{r}\Phi_0'$, $\sqrt{r}\Phi_1'$, $\Phi_1 \in L^2$; the integrals of the space-time angular momentum densities (4.15d) and dilational density (4.15e) if these and Φ_0 are L^2.

For (4.15f), we assume $r\Psi_0$, $r\Psi_1$, $r\Phi_0'$, $r\Phi_1'$, Φ_0, $\Phi_1 \in L^2$. The time derivative is

$$t^2 \Psi_1 \cdot \{-D\Phi_1 + (1-\beta)\, d\Psi_0\} + t|\Psi_1|^2 + r^2 \Psi_1 \cdot \{-D\Phi_1 + (1-\beta)\, d\Psi_0\}$$
$$- \beta t^2 \Psi_0 \cdot \left\{-\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta} \delta\Psi_1\right\} - \beta t |\Psi_0|^2$$
$$- \beta r^2 \Psi_0 \cdot \left\{-\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta} \delta\Psi_1\right\} + t^2 d\Phi_1 \cdot d\Psi_1 + t|d\Phi_1|^2$$
$$+ r^2 d\Phi_1 \cdot d\Psi_1 + \beta t^2 \delta\Phi_1 \cdot \delta\Psi_1 + \beta t |\delta\Phi_1|^2 + \beta r^2 \delta\Phi_1 \cdot \delta\Psi_1$$
$$- t^2 d\Phi_0 \cdot d\Psi_0 - t|d\Phi_0|^2 - r^2 d\Phi_0 \cdot d\Psi_0 - \beta t^2 \delta\Phi_0 \cdot \delta\Psi_0 - \beta t |\delta\Phi_0|^2$$
$$- \beta r^2 \delta\Phi_0 \cdot \delta\Psi_0 - 2t\beta R\Phi_0 \cdot \left\{-\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta} \delta\Psi_1\right\} - 2t\beta R\Psi_0 \cdot \Psi_0$$
$$- 2\beta R\Phi_0 \cdot \Psi_0 + 2tR\Phi_1 \cdot \{-D\Phi_1 + (1-\beta)\, d\Psi_0\}$$
$$+ 2tR\Psi_1 \cdot \Psi_1 + 2R\Phi_1 \cdot \Psi_1 - 2(1-\beta)\, tR\Phi_1 \cdot d\Psi_0$$

$$
\begin{aligned}
&-2(1-\beta)\,tR\Psi_1\cdot d\Phi_0-2(1-\beta)\,R\Phi_1\cdot d\Phi_0-(n-2k+2)\,\beta\Phi_0\cdot\Psi_0\\
&-(n-2k-2)\,\Phi_1\cdot\Psi_1-(n-2)\,\beta t\Phi_0\cdot\left\{-\frac{1}{\beta}D\Phi_0+\frac{1-\beta}{\beta}\delta\Psi_1\right\}\\
&-(n-2)\,\beta t\,|\Psi_0|^2-(n-2)\,\beta\Phi_0\cdot\Psi_0\\
&+(n-2)\,t\Phi_1\cdot\{-D\Phi_1+(1-\beta)\,d\Psi_0\}+(n-2)\,t\,|\Psi_1|^2\\
&+(n-2)\,\Phi_1\cdot\Psi_1-2\beta\Phi_1\cdot\left(\tilde{R}\wedge\left\{-\frac{1}{\beta}D\Phi_0+\frac{1-\beta}{\beta}\delta\Psi_1\right\}\right)\\
&-2\beta\Psi_1\cdot(\tilde{R}\wedge\Psi_0)+2\Phi_0\cdot\iota(R)\{-D\Phi_1+(1-\beta)\,d\Psi_0\}\\
&+2\Psi_0\cdot\iota(R)\,\Psi_1-2\Phi_0\cdot\iota(R)\,d\Psi_0-2\Psi_0\cdot\iota(R)\,d\Phi_0\\
&-2\beta\Phi_1\cdot d\iota(R)\,\Psi_1-2\beta\Psi_1\cdot d\iota(R)\,\Phi_1-(n-2)(1-\beta)\,t\Phi_1\cdot d\Psi_0\\
&-(n-2)(1-\beta)\,t\Psi_1\cdot d\Phi_0-(n-2)(1-\beta)\,\Phi_1\cdot d\Phi_0.
\end{aligned}
$$

The terms involving Ψ_0 only or Ψ_1 only are equivalent to 0 by (4.9). Terms involving Φ_0 only are

$$
\begin{aligned}
-t\,|d\Phi_0|^2&-\beta t\,|\delta\Phi_0|^2+2tR\Phi_0\cdot D\Phi_0+(n-2)\,t\Phi_0\cdot D\Phi_0\\
&\cong -t\,|d\Phi_0|^2-\beta t\,|\delta\Phi_0|^2+2t(Rd\Phi_0+d\Phi_0)\cdot d\Phi_0\\
&\quad+2\beta t(R\delta\Phi_0+\delta\Phi_0)\cdot\delta\Phi_0+(n-2)\,t\,|d\Phi_0|^2+(n-2)\,\beta t\,|\delta\Phi_0|^2
\end{aligned}
$$

by (4.10) and (4.11), which is equivalent to 0 by (4.9). The calculation for Φ_1 is the same.

Terms involving Φ_0 and Ψ_0 are

$$
\begin{aligned}
t^2\Psi_0\cdot D\Phi_0&+r^2\Psi_0\cdot D\Phi_0-t^2d\Phi_0\cdot d\Psi_0-r^2d\Phi_0\cdot d\Psi_0-\beta t^2\delta\Phi_0\cdot\delta\Psi_0\\
&-\beta r^2\delta\Phi_0\cdot\delta\Psi_0-2\beta R\Phi_0\cdot\Psi_0-2(n-k)\,\beta\Phi_0\cdot\Psi_0\\
&-2\beta\Phi_0\cdot\iota(R)\,d\Psi_0-2\Psi_0\cdot\iota(R)\,d\Phi_0,
\end{aligned}
$$

which, by (4.12) and (4.13), is equivalent to

$$
\begin{aligned}
-2\beta\Phi_0\cdot(d\iota(R)+\iota(R)\,d)\,\Psi_0&-2\beta R\Phi_0\cdot\Psi_0-2(n-k)\,\beta\Phi_0\cdot\Psi_0\\
&=-2\beta\Phi_0\cdot R\Psi_0-2(n-1)\,\beta\Phi_0\cdot\Psi_0-2\beta R\Phi_0\cdot\Psi_0\cong 0
\end{aligned}
$$

by (4.7) and (4.14). The argument for terms involving Φ_1 and Ψ_1 is similar.

Terms involving Φ_0 and Ψ_1 are

$$
\begin{aligned}
-2t(1-\beta)\,R\Phi_0\cdot\delta\Psi_1&-2(1-\beta)\,tR\Psi_1\cdot d\Phi_0\\
&-(n-2)(1-\beta)\,t\Phi_0\cdot\delta\Psi_1-(n-2)(1-\beta)\,t\Psi_1\cdot d\Phi_0
\end{aligned}
$$

$$\begin{aligned}&\cong -2(1-\beta)\, \iota(Rd\Phi_0 + d\Phi_0)\cdot \Psi_1\\&\quad - 2(1-\beta)\, \iota\Psi_1\cdot(-Rd\Phi_0 - (n-1)\, d\Phi_0)\\&\quad - 2(n-2)(1-\beta)\, \iota\Psi_1\cdot d\Phi_0 = 0.\end{aligned}$$

The terms involving Φ_1 and Ψ_0 vanish identically. Terms in Ψ_0 and Ψ_1 are equivalent to 0 by (4.12).

Finally, we list the terms involving Φ_0 and Φ_1:

$$\begin{aligned}&-2(1-\beta)\, R\Phi_1\cdot d\Phi_0 + 2\Phi_1\cdot(\tilde{R}\wedge D\Phi_0) - 2\Phi_0\cdot \iota(R)\, D\Phi_1\\&\quad - (n-2)(1-\beta)\, \Phi_1\cdot d\Phi_0\\&\cong -2(1-\beta)\, R\Phi_1\cdot d\Phi_0 + 2d\Phi_0\cdot\{d\iota(R) + \iota(R)\, d\}\, \Phi_1\\&\quad - 2\beta\{\delta\varepsilon(\tilde{R}) + \varepsilon(\tilde{R})\,\delta\}\, \Phi_0\cdot\delta\Phi_1 - (n-2)(1-\beta)\, \Phi_1\cdot d\Phi_0.\end{aligned}$$

Now $\delta\varepsilon(\tilde{R}) + \varepsilon(\tilde{R})\,\delta$ is the formal adjoint of L_R, which, on $(k-1)$-forms, is $-R-(n-k)$. Thus the above is

$$\begin{aligned}&-2(1-\beta)\, R\Phi_1\cdot d\Phi_0 + 2d\Phi_0\cdot(R+k)\, \Phi_1 - 2\beta(-R-n+k)\, \Phi_0\cdot\delta\Phi_1\\&\quad - (n-2)(1-\beta)\, \Phi_1\cdot d\Phi_0.\end{aligned}$$

By (4.10) and (4.7), $R\Phi_0\cdot\delta\Phi_1 = -d\Phi_0\cdot R\Phi_1 - (n-2)\, d\Phi_0\cdot\Phi_1$, so the above is equivalent to

$$[-(n-2k-2) + \beta(n-2k+2)]\, \Phi_1\cdot d\Phi_0.$$

This reduces to zero only if β has the distinguished value $(n-2k-2)/(n-2k+2)$.

By a calculation even more tedious than the above, the time derivative of (4.15g) is equivalent to 0, provided $r\Psi_0$, $r\Psi_1$, $r\Phi_0'$, $r\Phi_1'$, Φ_0, $\Phi_1\in L^2$. Again, the distinguished value for β is necessary. Thus we have:

THEOREM 5.1. *The integrals of* (4.15a) *and* (4.15b) *exist and are conserved by solutions of* (2.3) *if* Ψ_0, Ψ_1, Φ_0', $\Phi_1'\in L^2$ *at each fixed time. If* $\sqrt{r}\Psi_0$, $\sqrt{r}\Psi_1$, $\sqrt{r}\Phi_0'$, $\sqrt{r}\Phi_1'$, Φ_0, $\Phi_1\in L^2$ *at each fixed time, the integrals of* (4.15c)–(4.15e) *exist and are conserved. If* $r\Psi_0$, $r\Psi_1$, $r\Phi_0'$, $r\Phi_1'$, Φ_0, *and* $\Phi_1\in L^2$, *the integrals of* (4.15f) *and* (4.15g) *exist and are conserved.*

6. Modifications for the Nonlinear Equations

For Φ a k-form on $\mathbb{R}^{n-1}$, we set $u(\Phi) = |\Phi_1|^2 - |\Phi_0|^2$. The nonlinear system of Section 2, in evolution form, is

$$\begin{aligned} \dot{\Psi}_0 &= -\frac{1}{\beta} D\Phi_0 + \frac{1-\beta}{\beta} \delta\Psi_1 - \frac{\alpha}{\beta} u(\Phi)^{2/(n-2)} \Phi_0 \\ \dot{\Psi}_1 &= -D\Phi_1 + (1-\beta)\, d\Psi_0 - \alpha u(\Phi)^{2/(n-2)} \Phi_1. \end{aligned} \tag{6.1}$$

This is also conformally covariant in the sense of (2.2), and also has $(n+1)(n+2)/2$ independent conserved quantities.

For the energy, we proceed just as in Section 4, except that $-(\alpha/\beta)\, u(\Phi)^{2/(n-2)} \Phi_0$ must be added to W_0, and $-\alpha u(\Phi)^{2/(n-2)} \Phi_1$ to W_1 in the formal vector field defining time translation in Cauchy data space. The effect is to add

$$\begin{aligned} &-\alpha\langle X_0, u(\Phi)^{2/(n-2)} \Phi_0\rangle + \alpha\langle X_1, u(\Phi)^{2/(n-2)} \Phi_1\rangle \\ &\quad = \operatorname{Lin} \left\{ \frac{n-2}{2n} \alpha u(\Phi)^{n/(n-2)} \right\} \end{aligned}$$

to $\Omega((Z, W), (X, Y))$, and thus to add the quantity in brackets to E in (4.15a). (For the wave equation this is $((n-2)/2n)\, \alpha\phi^{2n/(n-2)}$.) For the integral of E to exist, we must add the conditions Φ_0, $\Phi_1 \in L^{2n/(n-2)}$. The computation of $\dot{E}$ proceeds as in Section 5, with two modifications: because $\dot{\psi}$ is given by (6.1) instead of (2.3), we get the extra terms

$$\Psi_1 \cdot (-\alpha u(\Phi)^{2/(n-2)} \Phi_1) - \beta\Psi_0 \cdot \left(-\frac{\alpha}{\beta} u(\Phi)^{2/(n-2)} \Phi_0 \right). \tag{6.2}$$

Because E has the extra term $E^+ = ((n-2)/2n)\, \alpha u(\Phi)^{n/(n-2)}$, we must also add to $\dot{E}$

$$\dot{E}^+ = \alpha u(\Phi)^{2/(n-2)} (\Phi_1 \cdot \Psi_1 - \Phi_0 \cdot \Psi_0). \tag{6.3}$$

The net correction, (6.2) plus (6.3), is identically 0.

The nonlinearity has no effect on the calculation of (4.15b) and (4.15c). $\dot{p}_i$ is changed by the exct divergence $-\partial_i E^+$, and the time derivatives of the angular momentum densities are changed by the exact divergence $-\partial_j(x^i E^+) + \partial_i(x^j E^+)$. Thus for $\int p_i$ to be conserved, we need the added assumptions Φ_0, $\Phi_1 \in L^{2n/(n-2)}$, while for the angular momenta, we need $r\,|\Phi_0|^{2n/(n-2)}$ and $r\,|\Phi_1|^{2n/(n-2)}$ to be integrable.

In calculating (4.15d), we add $-(\alpha/\beta)\, x^i u(\Phi)^{2/(n-2)} \Phi_0$ to W_0 and $-\alpha x^i u(\Phi)^{2/(n-2)} \Phi_1$ to W_1. The result is that (4.15d) is correct, with the new expression for E. For existence of the integral, we need $r|\Phi_0|^{2n/(n-2)}$ and $r|\Phi_1|^{2n/(n-2)}$ to be integrable. With this decay, it is easily seen that the integral is conserved. In calculating (4.15e), we add $-(\alpha/\beta)\,tu(\Phi)^{2/(n-2)}\,\Phi_0$ to W_0 and $-\alpha tu(\Phi)^{2/(n-2)}\,\Phi_1$ to W_1; (4.15e) is then correct with the new value of E. The correction to the time derivative simplifies to $-\mathrm{div}(E^+R)$, so the required decay assumption is the integrability of $r|\Phi_0|^{2n/(n-2)}$, $r|\Phi_1|^{2n/(n-2)}$.

In calculating (4.15f), we add $-(\alpha/\beta)(r^2+t^2)\,u(\Phi)^{2/(n-2)}\,\Phi_0$ to W_0 and $-\alpha(r^2+t^2)\,u(\Phi)^{2/(n-2)}\,\Phi_1$ to W_1. The result is that (4.15f) is still correct (with the new E). The correction to the time derivative reduces to $-2t\,\mathrm{div}(E^+R)$, so we add the decay assumptions $r|\Phi_0|^{2n/(n-2)}$, $r|\Phi_1|^{2n/(n-2)}\in L^1$.

For (4.15g), we argument W_0 by $2(\alpha/\beta)\,x^i tu(\Phi)^{2/(n-2)}\,\Phi_0$ and W_1 by $2\alpha x^i tu(\Phi)^{2/(n-2)}\,\Phi_1$. Equation (4.15g) is then correct with the new E. Calculations similar to the above show that the integral is conserved under the same decay assumptions as (4.15f). We have:

THEOREM 6.1. *Let*

$$E=\frac{1}{2}|\Psi_1|^2-\frac{\beta}{2}|\Psi_0|^2+\frac{1}{2}|d\Phi_1|^2+\frac{\beta}{2}|\delta\Phi_1|^2-\frac{1}{2}|d\Phi_0|^2 \\ -\frac{\beta}{2}|\delta\Phi_0|^2+\frac{n-2}{2n}\alpha\{|\Phi_1|^2-|\Phi_0|^2\}^{n/(n-2)}.$$

Then the integrals of E and (4.15b) *exist and are conserved by solutions of* (6.1) *provided Ψ_0, Ψ_1, Φ_0', $\Phi_1'\in L^2$ and Φ_0, $\Phi_1\in L^{2n/(n-2)}$ at each fixed time. The integrals of* (4.15c)–(4.15e) (*with the above value for E*) *exist and are conserved if $\sqrt{r}\Psi_0$, $\sqrt{r}\Psi_1$, $\sqrt{r}\Phi_0'$, $\sqrt{r}\Phi_1'$, Φ_0, $\Phi_1\in L^2$, and $r|\Phi_0|^{2n/(n-2)}$, $r|\Phi_1|^{2n/(n-2)}$ are integrable, at each fixed time. The integrals of* (4.15f) *and* (4.15g) (*with the above value for E*) *exist and are conserved if $r\Psi_0$, $r\Psi_1$, $r\Phi_0'$, $r\Phi_1'$, Φ_0, $\Phi_1\in L^2$ and $r|\Phi_0|^{2n/(n-2)}$, $r|\Phi_1|^{2n/(n-2)}$ are integrable at each fixed time.*

7. DISCUSSION

Remark 7.1. (4.1) actually generalizes the usual symplectic structure for the Maxwell equations, as well as that for the wave equation. The Maxwell equations $\delta^{(n)}d^{(n)}\Phi=0$ on an $(n-2)/2$-form Φ are *gauge-invariant* in the sense that any exact form $d^{(n)}\xi$ may be added to Φ without affecting the equations (or even the field strengths $\Omega=d^{(n)}\Phi$). *Solutions* are gauge equivalence classes of Φ's. The Cauchy problem is not well-posed without gauge reduction to the case $\Phi_0 = 0$ (the *temporal gauge*). This is accomplished by taking any time-dependent $(k-1)$-form α on $\mathbb{R}^{n-1}$ with $\dot{\alpha} = \Phi_0$ and subtracting $d^{(n)}\alpha$ to obtain Φ':

$$\Phi'_0 = 0, \qquad \Phi'_1 = \Phi_1 - d\alpha.$$

The equations then read (setting $\Psi'_1 = \dot{\Phi}'_1$)

$$\delta\Psi'_1 = 0, \qquad \dot{\Psi}'_1 + \delta d\Phi'_1 = 0.$$

Cauchy data space is the Cartesian product of the space of all Φ'_1 with the space of all coclosed Ψ'_1 ($\delta\Psi'_1 = 0$). The usual symplectic structure is

$$\Omega_M((Z', W'), (X', Y')) = \langle Z', Y'\rangle - \langle X', W'\rangle, \tag{7.1}$$

where Z' is the Φ'_1 component and W the Ψ'_1 component of the first vector field, etc. If we imagine (Z', W') as coming from some general pair of k-forms $(Z = dt \wedge Z_0 + Z_1,\ W = dt \wedge W_0 + W_1)$ through reduction to the temporal gauge, $Z' = Z_1 - d\zeta$, $\dot{\zeta} = Z_0$, $W' = W_1 - dZ_0$, and similarly for (X', Y') ($\dot{\xi} = X_0$), (7.1) becomes

$$\begin{aligned}\Omega_M&((Z, W), (X, Y))\\ &= \langle Z_1 - d\zeta, Y_1 - dX_0\rangle - \langle X_1 - d\xi, W_1 - dZ_0\rangle.\end{aligned} \tag{7.2}$$

The difference between the right sides of (4.1) (with $\beta = 0$ as is appropriate to $k = (n-2)/2$) and (7.2) is

$$\begin{aligned}-\langle d\zeta, Y_1 - dX_0\rangle + \langle d\xi, W_1 - dZ_0\rangle &= -\langle d\zeta, Y'\rangle + \langle d\xi, W'\rangle\\ &= -\langle \zeta, \delta Y'\rangle + \langle \xi, \delta W'\rangle = 0,\end{aligned}$$

since the Ψ'-component of a vector in Cauchy data space is coclosed.

The energy is not given by (4.15a) however (we divide by β in the calculation leading to (4.15a)). The conserved quantities for the Maxwell equations may be calculated using the same heuristic as above, though we do not give the calculations here. These are gauge invariant, and depend only on the field strengths Ω and not directly on Φ. As shown in [1], the same nonlinear term as above may be added to the equation $\delta d\Phi = 0$; the conserved quantities for the nonlinear equation *do* depend directly on Φ. (The nonlinear equations seem not to have any gauge invariance features analogous to that for the Maxwell equations.)

Remark 7.2. Unlike the wave equation [3], the Maxwell and Yang–Mills systems [2] and the nonlinear wave equation $\Box\phi + \phi^{(n+2)/(n-2)} = 0$ [5], the equations treated here are not put in the form $\ddot{\Phi} + G(\Phi) = 0$. Indeed, it seems that there is no natural change of variable in phase space which puts the equations in this form. We can, however, write down "Darboux coordinates" for the symplectic form (4.1). If we set

$$q=(q_0, q_1)=(\Phi_0, \Phi_1),$$
$$p=(p_0, p_1)=(\beta(\Psi_0+\delta\Phi_1), \Psi_1-d\Phi_0),$$

the equations (2.3) become

$$\dot{q}_0=\frac{1}{\beta}p_0-\delta q_1,$$

$$\dot{q}_1=p_1+dq_0,$$

$$\dot{p}_0=\frac{1}{\beta}\delta p_1-d\delta q_0,$$

$$\dot{p}_1=-dp_0-\delta dq_1.$$

If (Q, P), (Q', P') are the (q, p)-coordinates of two formal vector fields (Z, W) and (X, Y), then

$$\Omega((Z, W), (X, Y))=(\langle Q_1, P_1'\rangle-\langle Q_0, P_0'\rangle)-(\langle Q_1', P_1\rangle-\langle Q_0', P_0\rangle).$$

Remark 7.3. The referee has pointed out that the equations (2.3), in addition to being Hamiltonian, are the Euler–Lagrange equations for the action integral

$$I(\Phi)=-\int_M \tfrac{1}{2}(\Phi, \tilde{\Box}\Phi)\, d^n x,$$

where $d^n x$ is the measure on the n-dimensional Minkowski space M, and the inner product $(\ ,\ )$ is given by

$$(\Phi, \Phi')=\Phi_1\cdot\Phi_1'-\Phi_0\cdot\Phi_0'.$$

In fact, this is the case for these equations on more general manifolds M (see [1]). The action integral is invariant under the multiplier action (2.1) of the conformal group. (See [1] for a more detailed account of the conformal action. The proof of the invariance of the action is carried out in [1, Theorem 4.4] for compact Riemannian M, but the argument is the same for a general pseudo-Riemannian M provided $I(\Phi)$ exists.)

Thus the conserved quantities arise from the variational principle via Noether's Theorem, and may be calculated using the Lagrangian rather than Hamiltonian formalism. The Hamiltonian E of (4.3) is given by the infinite-dimensional analogue of $E=p\dot{q}-L$, where L is the action density and p and

q are as in Remark 7.2. Calculating formally, assuming that Φ and its derivatives vanish at timelike and spacelike infinity,

$$\begin{aligned} I(\Phi) &= -\tfrac{1}{2}\int_{-\infty}^{\infty} \{\langle \Phi_1, \ddot{\Phi}_1 + D\Phi, -(1-\beta)\, d\dot{\Phi}_0\rangle \\ &\qquad - \langle \Phi_0, \beta\ddot{\Phi}_0 + D\Phi_0 - (1-\beta)\,\delta\dot{\Phi}_1\rangle\}\, dt \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} L\,dt, \end{aligned}$$

where

$$\begin{aligned} L = {} & \frac{1}{2}\langle \Psi_1, \Psi_1\rangle - \frac{1}{2}\langle d\Phi_1, d\Phi_1\rangle - \frac{\beta}{2}\langle \delta\Phi_1, \delta\Phi_1\rangle \\ & - \frac{\beta}{2}\langle \Psi_0, \Psi_0\rangle + \frac{1}{2}\langle d\Phi_0, d\Phi_0\rangle + \frac{\beta}{2}\langle \delta\Phi_0, \delta\Phi_0\rangle \\ & - \langle d\Phi_0, \Psi_1\rangle - \beta\langle \delta\Phi_1, \Psi_0\rangle. \end{aligned}$$

The analogue of $p\dot{q}$ is

$$\begin{aligned} & \langle p_1, \Psi_1\rangle - \langle p_0, \Psi_0\rangle \\ & \quad = \langle \Psi_1, \Psi_1\rangle - \langle d\Phi_0, \Psi_1\rangle - \beta\langle \Psi_0, \Psi_0\rangle - \beta\langle \delta\Phi_1, \Psi_0\rangle. \end{aligned}$$

Remark 7.4. The conserved quantities for the wave equation, the special case $k = 0$ of the above, appear in [6].

Remark 7.5. The main obstacle to applying the standard technique of using the energy and nth inversional quantity to obtain decay estimates (as in [2]) for the new systems is the nonpositivity of the energy. It is not clear whether this difficulty can be overcome.

References

1. T. Branson, Conformally covariant equations on differential forms, *Comm. P.D.E.* **7** (1982), 393–431.
2. R. Glassey and W. Strauss, Decay of classical Yang–Mills fields, *Comm. Math. Phys.* **65** (1979), 1–13.
3. C. Morawetz, The limiting amplitude principle, *Comm. Pure Appl. Math.* **15** (1962), 349–361.
4. I. E. Segal, "Mathematical Cosmology and Extragalactic Astronomy," Academic Press, New York, 1976.
5. W. Strauss, La decroissance asymptotique des solutions des équations d'onde nonlinéaires, *C. R. Acad. Sci.* **256** (1963), 2749–2750.
6. W. Strauss, Nonlinear invariant wave equations, *in* Lecture Notes in Physics No. 73, pp. 197–249, Springer-Verlag, New York, 1978.
7. F. Warner, "Foundations of Differentiable Manifolds and Lie Groups," Scott, Foresman, Glenview, Ill., 1971.